The Jacobi Geometry of Plane Parametrized Curves and Associated Inequalities

Mircea Crasmareanu

Abstract Firstly, we introduce a new frame and a new curvature function for a fixed parametrization r of a plane curve C. This new frame is called *Jacobi* since it involves the rotation with the first two Jacobi elliptic functions of the usual Frenet frame. The Jacobi-curvature involves only the third Jacobi elliptic function w and is computed for some remarkable examples; the inequalities satisfied by w imply inequalities for the Jacobi-curvature. Secondly, we introduce a whole family of new parametrizations r_ρ for C with $r = r_{\rho=0}$. The expression of r_ρ involves an integral containing the curvature function k of r, and all r_ρ have the same curvature.

Keywords Plane parametrized curve · Jacobi elliptic functions · Inequalities · Jacobi-curvature · Jacobi mate

1 Introduction

The delightful note [1] of Bishop proposes a new frame, as alternative to the classical frame of Frenet, for the study of curves. Following this path, in an almost half of a century, some new frames are considered, especially for space curves; see, for example, the papers [9] and [10]. In order to consider the case of a plane curve C, we introduce in the paper [3] a deformation (following the Masur terminology from [8]) called *flow-frame* since it is the rotated version of the Frenet frame, the rotation angle being exactly the time t of a current point of C. It follows naturally a new curvature, called *flow-curvature*. We point out that some other curvature functions are defined in the paper [7].

In the present work we firstly generalize this construction by defining the Jacobi-frame of C using the well-known Jacobi elliptic functions u, v, w. These functions

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are defined through a modulus ρ, and the vanishing of ρ implies $u = \cos$ and $v = \sin$. Correspondingly, this new frame defines a curvature, called *Jacobi* by us from natural reasons and denoted k_J. Hence, the main theoretical result of this note is the computation of k_J and a comparison with the usual curvature k, as well as a relationship with the flow-curvature k_f. As is usually, we focus then on examples, with a special view toward periodicity induced by the periodicity of the third Jacobi function w.

Secondly, we generalize a given arc-length parametrization $r = r(s)$ of C. The main tool of this new approach is an integral involving the ratio of k and w, and it is worth to remark that all new parametrizations are also arc-length and with the same curvature k. In fact, the initial r corresponds to $\rho = 0$. The difficulties in working with general Jacobi elliptic functions force us to restrict the examples to the circles with center in the origin of $\mathbb{R}^2$.

The contents are as follows. The next section reviews the flow-frame and the flow-curvature as starting point for our generalization. The main section, namely 2, concerns with the new frame and the new curvature, computed in Proposition 2.2 and calculated for some examples. Also, we point out an extension of the notion of *Jacobi-frame* to space curves. In the last section we construct the family of new parametrizations r_ρ preserving the natural parameter s and the curvature k.

We finish this introduction by pointing out that the efficiency of the Jacobi generalization is already proved by our study [5]. As potential applications we consider that the Computer Design and the Machine Learning can benefit from such new tools.

2 The Flow-Curvature of a Plane Parametrized Curve

Fix an open interval $I \subseteq \mathbb{R}$, and consider $C \subset \mathbb{R}^2$ a regular parametrized curve of equation:

$$C : r(t) = (x(t), y(t)) = x(t)\bar{i} + y(t)\bar{j}, \quad \|r'(t)\| > 0, \quad t \in I. \tag{2.1}$$

The ambient setting $\mathbb{R}^2$ is a Euclidean vector space with respect to the canonical inner product:

$$\langle u, v\rangle = x^1y^1 + x^2y^2, \quad u = (x^1, x^2) \in \mathbb{R}^2, \quad v = (y^1, y^2) \in \mathbb{R}^2,$$
$$0 \leq \|u\|^2 = \langle u, u\rangle. \tag{2.2}$$

The infinitesimal generator of the rotations in $\mathbb{R}^2 = \mathbb{C}$ is the linear vector field, called *angular*:

$$\xi(u) := -x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2}, \quad \xi(u) = i \cdot u = i \cdot (x^1 + ix^2), \quad i = \sqrt{-1}. \tag{2.3}$$

It is a complete vector field with integral curves the circles $\mathcal{C}(O, r)$:

$$\begin{cases} \gamma^{\xi}_{u_0}(t) = (u_0^1 \cos t - u_0^2 \sin t, u_0^1 \sin t + u_0^2 \cos t) = R(t) \cdot \begin{pmatrix} u_0^1 \\ u_0^2 \end{pmatrix}, \quad t \in \mathbb{R}, \\ r = \|u_0\| = \|(u_0^1, u_0^2)\|, \quad R(t) := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SO(2) = S^1, \end{cases} \tag{2.4}$$

and since the rotations $R(t)$ are isometries of the Riemannian metric $g_{can} = dx^2 + dy^2 = |dz|$, it follows that ξ is a Killing vector field of the Riemannian manifold $(\mathbb{R}^2, g_{can})$. The first integrals of ξ are the Gaussian functions, i.e., multiples of the square norm: $f_\alpha(x, y) = \alpha(x^2 + y^2)$, $\alpha \in \mathbb{R}$.

The Frenet apparatus of the curve C is provided by the Frenet frame $\{T, N\}$ and its curvature function k:

$$\begin{cases} T(t) = \frac{r'(t)}{\|r'(t)\|} \in S^1, \quad N(t) = i \cdot T(t) = \frac{1}{\|r'(t)\|}(-y'(t), x'(t)) \in S^1, \\ k(t) = \frac{1}{\|r'(t)\|}\langle T'(t), N(t)\rangle = \frac{1}{\|r'(t)\|^3}\langle r''(t), ir'(t)\rangle \\ = \frac{1}{\|r'(t)\|^3}[x'(t)y''(t) - y'(t)x''(t)]. \end{cases} \tag{2.5}$$

The starting point in defining a new frame is the identity:

$$\frac{d}{dt}R(t) = R\left(t + \frac{\pi}{2}\right) = R(t)R\left(\frac{\pi}{2}\right) = R\left(\frac{\pi}{2}\right)R(t), \tag{2.6}$$

and remark that the Frenet equations can be unified by means of the column matrix $\mathcal{F}(t) = \begin{pmatrix} T \\ N \end{pmatrix}(t)$ as

$$\frac{d}{dt}\mathcal{F}(t) = \|r'(t)\|k(t)R\left(-\frac{\pi}{2}\right)\mathcal{F}(t). \tag{2.7}$$

In a previous paper, namely [3], we have defined a new frame and correspondingly a new curvature function for C:

Definition 2.1 The *flow-frame* of C consists in the pair of unit vectors $(E_1^f(t), E_2^f(t)) \in T^2 := S^1 \times S^1$ given by

$$\mathcal{E}(t) := \begin{pmatrix} E_1^f \\ E_2^f \end{pmatrix}(t) = R(t)\mathcal{F}(t) = \begin{pmatrix} \cos t T(t) - \sin t N(t) \\ \sin t T(t) + \cos t N(t) \end{pmatrix} \tag{2.8}$$

the letter f being the initial of the word "flow." The *flow-curvature* of C is the smooth function $k_f : I \to \mathbb{R}$ given by *the flow-equations*:

$$\frac{d}{dt}\mathcal{E}(t) = \|r'(t)\|k_f(t)R\left(-\frac{\pi}{2}\right)\mathcal{E}(t). \tag{2.9}$$

Hence, the main result of the cited work is the following:

Proposition 2.2 *The expression of the flow-curvature is*

$$k_f(t) = k(t) - \frac{1}{\|r'(t)\|} < k(t). \tag{2.10}$$

Proof We have directly in the flow-frame

$$\|r'(t)\|k_f(t)R\left(-\frac{\pi}{2}\right) = R\left(t+\frac{\pi}{2}\right)R(-t) + \|r'(t)\|k(t)R(t)R\left(-\frac{\pi}{2}\right)R(-t), \tag{2.11}$$

and the conclusion follows. □

3 The Jacobi-Curvature of a Plane Parametrized Curve

Fix now the real number $\rho \in (-1, 1)$ as *the modulus* for the differential system [6, p. 130]:

$$\begin{cases} \frac{du}{dt} = -wv, \quad u(0) = 1, \\ \frac{dv}{dt} = wu, \quad v(0) = 0, \\ \frac{dw}{dt} = -\rho^2 uv, \quad w(0) = 1. \end{cases} \tag{3.1}$$

Recall that its solutions are called *Jacobi elliptic functions* and there are usually denoted $cn(\cdot, \rho)$, $sn(\cdot, \rho)$, respectively, $dn(\cdot, \rho)$; we prefer the simple notation used above. As solutions of the ODE system (3.1) these functions satisfy two remarkable identities:

$$u^2 + v^2 = 1, \quad \rho^2 v^2 + w^2 = 1. \tag{3.2}$$

Also, both functions $u(\cdot)$ and $v(\cdot)$ are periodic with $L = 4\tilde{L}$ for [6, p. 131]:

$$\tilde{L} = \tilde{L}(\rho) := \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-\rho^2 s^2)}}, \tag{3.3}$$

while w is periodic of period $2\tilde{L}$. In particular, $\tilde{L}(0) = \arcsin s|_0^1 = \frac{\pi}{2}$ for the usual trigonometrical functions $cn(\cdot, 0) = \cos(\cdot)$ and $sn(\cdot, 0) = \sin(\cdot)$. The *complementary modulus* is $\rho' := \sqrt{1-\rho^2} \in (0, 1]$, and the third Jacobi function is bounded by

$$0 < \rho' \leq w(t) \leq 1. \tag{3.4}$$

The *self-complementary case* $\rho' = \rho$ is provided by $\rho = \frac{1}{\sqrt{2}}$ and being in the interval (0, 1) is the eccentricity of an ellipse, called *self-complementary* and studied in [2]. The picture of the function $w = w(\cdot, \rho = \frac{1}{\sqrt{2}})$ is below with the half-period:

$$\tilde{L}\left(\rho = \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}\Gamma(1/4)\Gamma(1/2)}{4\Gamma(3/4)} \simeq 1.85407.$$

Following the path of the first section we introduce a new frame and a new curvature function for the given curve (Fig. 1):

Definition 3.1 The *Jacobi-frame* of C consists in the pair of unit vectors $(E_1^J(t), E_2^J(t)) \in T^2$ given by

$$\begin{cases} \mathcal{E}^J(t) := \begin{pmatrix} E_1^J \\ E_2^J \end{pmatrix}(t) = R^J(t)\mathcal{F}(t) = \begin{pmatrix} u(t)T(t) - v(t)N(t) \\ v(t)T(t) + u(t)N(t) \end{pmatrix}, \\ R^J(t) := \begin{pmatrix} u(t) & -v(t) \\ v(t) & u(t) \end{pmatrix} \in SO(2) = S^1. \end{cases} \tag{3.5}$$

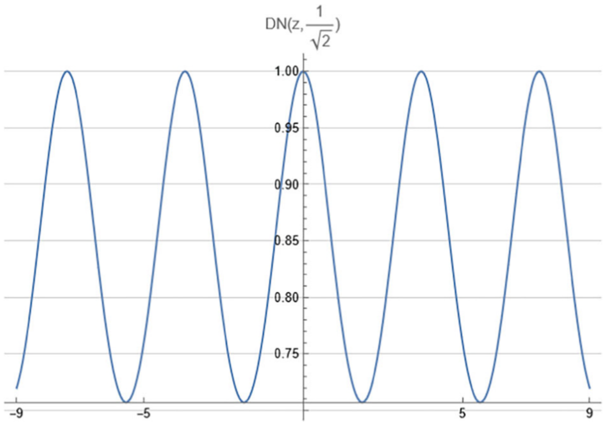

Fig. 1 The Jacobi function $w = w(t, \rho = 1/\sqrt{2}), \quad t \in (-9, 9)$

The *Jacobi-curvature* of C is the smooth function $k_J : I \to \mathbb{R}$ given by *the Jacobi-frame equations*:

$$\frac{d}{dt}\mathcal{E}^J(t) = \|r'(t)\|k_J(t)R\left(-\frac{\pi}{2}\right)\mathcal{E}^J(t). \tag{3.6}$$

It follows now the main result, with a similar proof as above:

Proposition 3.2 *The expression of the Jacobi-curvature is*

$$k_J(t) = k(t) - \frac{w(t)}{\|r'(t)\|} \in \left[k_f(t), k(t) - \frac{\rho'}{\|r'(t)\|} < k(t)\right]. \tag{3.7}$$

Remark 3.3

(i) If we use Eq. (2.8) with R replaced by $R \circ \Omega$ to define the notion of Ω-frame for the plane curve C, then the corresponding Ω-curvature of the plane curve C is

$$k_\Omega(t) = k(t) - \frac{\Omega'(t)}{\|r'(t)\|}, \tag{3.8}$$

and the curves in polar coordinates with vanishing Ω-curvature are provided by

$$\rho(t) = Re^{\int_{t_0}^{t} \cot[\Omega(u)-u+C]du}, \quad R > 0, \quad C \in \mathbb{R}. \tag{3.9}$$

The flow-curvature corresponds to the identity map $\Omega = 1_{\mathbb{R}}$, while the Jacobi-curvature corresponds to the function $\Omega = W := \int w$. This last function is usually called *amplitude*, and we supposed to be *strictly positive*.

(ii) It is well known the identity:

$$t = \int_0^{W(t)} \frac{d\xi}{\sqrt{1-\rho^2 \sin^2 \xi}},$$

and then we have the function $W \to t(W)$. The first two Jacobi differential equations become

$$\frac{du}{dW} = -v \circ t(W), \quad \frac{dv}{dW} = u \circ t(W),$$

which are similar to the differential equations satisfied by the trigonometrical functions $\cos$ and $\sin$.

(iii) Let $s \in (0, L(C) > 0)$ be a natural parameter for the curve C, i.e., $\|r'(s)\| = 1$ for all s. Here, $L(C)$ is the length of the curve. Let also $K = K(s)$ be *the structural angle* of C, i.e., $k = \frac{dK}{ds}$. Then k_J is a derivative:

$$k_J(s) = (K - W)'(s).$$

(iv) Suppose now that the curve C is in the space $\mathbb{R}^3$ and is bi-regular, i.e., $\|r'(t) \times r''(t)\| > 0$ for all $t \in I$; hence it has the Frenet frame (T, N, B) and the pair (curvature, torsion) = $(k > 0, \tau)$. We define its *Jacobi-frame* as

$$\begin{pmatrix} T \\ E_2^J \\ E_3^J \end{pmatrix}(t) := \begin{pmatrix} 1 & 0_2(h) \\ 0_2(v) & R^J(t) \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \quad 0_2(h) := (0, 0), \quad 0_2(v) := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.10}$$

and then, its matrix moving equation is

$$\frac{d}{dt}\begin{pmatrix} T \\ E_2^J \\ E_3^J \end{pmatrix}(t) = \|r'(t)\| \begin{pmatrix} 0 & k_J^2(t) & k_J^3(t) \\ -k_J^2(t) & 0 & \tau_J(t) \\ -k_J^3(t) & -\tau_J(t) & 0 \end{pmatrix} \begin{pmatrix} T \\ E_2^J \\ E_3^J \end{pmatrix}(t). \tag{3.11}$$

A similar computation yields

$$k_J^2(t) = k(t)u(t), \quad k_J^3(t) = k(t)v(t), \quad \tau_f(t) = \tau(t) - \frac{w(t)}{\|r'(t)\|} < \tau(t). \tag{3.12}$$

□

From now on we focus on computing some relevant examples:

Example 3.4

(i) If C is the line $r_0 + tu, t \in \mathbb{R}$, with the vector $u \neq \bar{0} = (0, 0)$, then k_J is periodic with the period $2\tilde{L}$ since

$$k_J(t) = -\frac{w(t)}{\|u\|} \in \left[k_f(t) = -\frac{1}{\|u\|}, -\frac{\rho'}{\|u\|}\right]. \tag{3.13}$$

In particular, if u is a unit vector, then $k_J(t) = -w(t) \in [-1, -\rho' < 0]$.

(ii) The circle $\mathcal{C}(O, R)$ with the usual parametrization $r(t) = Re^{it}$ has

$$k_J(t) = \frac{1 - w(t)}{R} \in \left[k_f = 0, \frac{1 - \rho'}{R}\right] \tag{3.14}$$

again a $2\tilde{L}$-periodic k_J curvature. Also, it follows a geometrical interpretation of the third Jacobi function: w is the function $1 - k_J$ of the unit circle S^1.

□

Example 3.5 The involute of the unit circle S^1 is

$$C : r(t) = (\cos t + t \sin t, \sin t - t \cos t) = (1 - it)e^{it}, \quad t \in (0, +\infty). \tag{3.15}$$

A direct computation gives

$$r'(t) = (t \cos t, t \sin t) = te^{it}, \quad \|r'(t)\| = t, \quad k(t) = \frac{1}{t} > 0,$$
$$k^J(t) = \frac{1 - w(t)}{t}. \tag{3.16}$$

The parametrization (3.15) suggests the following generalization; we call *Jacobi-involute of* S^1 the curve:

$$C_\rho : r_\rho(t) = (u(t) + W(t)v(t), v(t) - W(t)u(t)) \tag{3.17}$$

with

$$r'_\rho(t) = W(t)w(t)(u(t), v(t)), \quad \|r'_\rho(t)\| = W(t)w(t). \tag{3.18}$$

Finally, we obtain its curvatures:

$$k_\rho(t) = \frac{1}{W(t)}, \quad k_{\rho J} \equiv 0 \tag{3.19}$$

by recalling the hypothesis from Remark 3.3 (i) that $W(t) > 0$; hence C_ρ is a Jacobi-flat curve. The length of the curve $C_\rho|_{(0,L)}$ is

$$Length(C_\rho|_{(0,L)}) = \int_0^L W(t)W'(t)dt = \frac{W^2(L)}{2} = \frac{\pi^2}{8}. \tag{3.20}$$

□

Example 3.6 Recall that for $R > 0$ the cycloid of radius R has the equation

$$C : r(t) = R(t - \sin t, 1 - \cos t) = R[(t, 1) - e^{i(\frac{\pi}{2} - t)}], \quad t \in \mathbb{R}. \tag{3.21}$$

We have immediately

$$r'(t) = R(1-\cos t, \sin t) = R[(1,0) - e^{it}],\ \|r'(t)\| = 2R|\sin\frac{t}{2}|,\ k(t) = -\frac{1}{4R|\sin\frac{t}{2}|}, \tag{3.22}$$

and then we restrict our definition domain to $(0, \pi)$. It follows

$$k_f(t) = -\frac{3}{4R\sin\frac{t}{2}} < 0, \quad k_J(t) = -\frac{1+2w(t)}{4R\sin\frac{t}{2}}. \tag{3.23}$$

□

The expression of k_J suggests to define *the Jacobi-cycloid* as being the regular curve C defined on $(0, 2\tilde{L})$ whose Frenet curvature k is

$$k(t) = -\frac{1+2w(t)}{4Rv(\frac{t}{2})} \tag{3.24}$$

since the fundamental theorem of plane curves assures the existence of such a curve.

□

4 The Jacobi Mates of an Arc-Length Parametrization

Suppose again that the given parametrization is an arc-length one: $r'(s) \in S^1$. Here, $L(C)$ is the length of the curve. Recall, after Remark 3.3(iii), the function $K : (0, L(C)) \to \mathbb{R}$ as the antiderivative of the curvature function $k = k(s)$. Then the fundamental theorem of plane curves states that the velocity vector field is given by

$$r'(s) = (-\sin K(s), \cos K(s)) = (\cos\theta(s), \sin\theta(s)), \quad \theta(s) := \frac{\pi}{2} + K(s). \tag{4.1}$$

Sometimes, the function θ is called *the structural angle* of the curve C since $\theta' = k$.

An immediate application of the above relation is the fact that the defining functions x, y of the parametrization r of C satisfy the third-order differential equation [4]:

$$kU''' - k'U'' + k^3U' = 0. \tag{4.2}$$

Hence, the aim of this section is to define a generalization of (4.1). Following the path of the previous section, we introduce a new antiderivative for the given curve:

$$K^{\rho}(s) := \int_0^s \frac{k(t)}{w(t)} dt. \tag{4.3}$$

This integral can be considered as the antiderivative of k with *the weight* $\frac{1}{w} > 0$, and the relationship with K is

$$K^{\rho}(s) = \frac{K}{w}|_0^s - \rho^2 \int_0^s \frac{K(t)u(t)v(t)}{w^2(t)} dt = \frac{K(s)}{w(s)} - \rho^2 \int_0^s \frac{K(t)u(t)v(t)}{w^2(t)} dt. \tag{4.4}$$

Also, if k is strictly positive (e.g., C is a convex curve), then we have the inequalities:

$$K \leq K^{\rho} \leq \frac{K}{\rho'}. \tag{4.5}$$

Definition 4.1 *The Jacobi mate* of the arc-length parametrization r is the function $r_{\rho} : (0, L(C)) \to \mathbb{R}^2$ with the derivative

$$r'_{\rho}(s) := (-v(K^{\rho}(s)), u(K^{\rho}(s))). \tag{4.6}$$

Remark 4.2

(i) The curvature of r_{ρ} is also k since the acceleration function for r_{ρ} is

$$r''_{\rho}(s) := (-k(s)u(K^{\rho}(s)), -k(s)v(K^{\rho}(s))). \tag{4.7}$$

Hence, r_{ρ} is a re-parametrization of the same curve C, and its components x^{ρ}, y^{ρ} satisfy the same ODE (3.2).

(ii) The function $r_{\frac{1}{\sqrt{2}}}$ can be called *the self-complementary parametrization* of C.

□

Due to the complexity of computations we restrict to a single relevant example:

Example 4.3 The circle $\mathcal{C}(O, R)$ has the usual arc-length parametrization $r(s) = Re^{i\frac{s}{R}}$ for $s \in (0, 2\pi R)$. We need the function

$$\tilde{W}(s) := \int_0^s \frac{dt}{w(t)} = \frac{1}{i\rho'} \ln \frac{u(s) + i\rho' v(s)}{w(s)}, \tag{4.8}$$

and hence, $K^{\rho} = \frac{\tilde{W}}{R}$. Then the Jacobi mate is r_{ρ} with the derivative

$$r'_{\rho}(s) := \left(-v\left(\frac{\tilde{W}(s)}{R}\right), u\left(\frac{\tilde{W}(s)}{R}\right)\right). \tag{4.9}$$

Indeed, the case $\rho = 0$ recasts the usual r since then $\rho' = 1$, and the Euler formula gives $\tilde{W}(s) = s$. □

5 Conclusions

The Jacobi elliptic functions permit us to move beyond classical confines and provide us with a framework in which we generalize some usual notions of the differential geometry of plane curves.

6 Chapter Information

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